Int. J. Heat Mass Transfer. Vol. 14, pp. 1531–1534. Pergamon Press 1971. Printed in Great Britain

SHORTER COMMUNICATION

LIMIT OF PURE CONDUCTION FOR UNSTEADY FREE CONVECTION ON A VERTICAL PLATE

K. NANBU

Institute of High Speed Mechanics, Tohoku University, Sendai, Japan

(*Received* 8 *October* 1970)

NOMENCLATURE

erfc (η), complementary error function $= \dfrac{2}{\sqrt{\pi}} \displaystyle\int_{\eta}^{\infty} e^{-\lambda^2} d\lambda$;

g, acceleration of gravity;

Gr_x, local Grashof number $= g\beta x^3 \Delta T/\nu^2$

h, local heat-transfer coefficient;

i^n erfc (η), complementary error function integrated n times $= \int_{\eta}^{\infty} i^{n-1}\,\mathrm{erfc}(\lambda)\, d\lambda$, $i^0\,\mathrm{erfc}(\eta) = \mathrm{erfc}(\eta)$;

k, thermal conductivity of fluid;

Nu_x, local Nusselt number $= hx/k$;

Pr, Prandtl number $= \nu/\alpha$;

t, time;

T, temperature;

T_w, temperature of wall;

T_∞, temperature at infinity (also, initial temperature of fluid);

T^*, $= \tau/X^{\frac{1}{2}}$;

u, component of velocity in x-direction;

U, $= u/(\nu g\beta\Delta T)^{\frac{1}{3}}$;

U^*, $= U/X^{\frac{1}{2}} = \partial\phi/\partial\zeta$;

v, component of velocity in y-direction;

V, $= v/(\nu g\beta\,\Delta T)^{\frac{1}{3}}$;

x, vertical distance from bottom of plate;

X, $= x(g\beta\,\Delta T/\nu^2)^{\frac{1}{3}}$;

y, horizontal distance from plate;

Y, $= y(g\beta\,\Delta T/\nu^2)^{\frac{1}{3}}$.

Greek symbols

α, thermal diffusivity of fluid;

β, thermal expansion coefficient;

ΔT, $= T_w - T_\infty$;

ζ, $= y/2\sqrt{(\nu t)}$;

η, $= y/2\sqrt{(\alpha t)} = Pr^{\frac{1}{2}}\zeta$;

θ, $= (T - T_\infty)/(T_w - T_\infty)$;

ν, kinematic viscosity of fluid;

τ, $= t(g\beta\Delta T)^{\frac{2}{3}}/\nu^{\frac{1}{3}}$;

ϕ, function introduced in equation (6),

ψ, stream function.

INTRODUCTION

THE PROBLEM of unsteady laminar free convection on a vertical plate has been studied extensively [1]. For a plate of semi-infinite length, Sugawara and Michiyoshi [2] treated a step-function change in a wall temperature by using a method of successive approximations. The same problem was studied by Siegel [3], and Hellums and Churchill [4, 5]. The former employed Kármán–Pohlhausen method and the latter finite-difference method. As pointed out in the above investigations, heat transfer at an early stage is by pure conduction: the convective heat transfer in the ordinary sense has not started, and the temperature and velocity fields for the semi-infinite plate are the same as for the doubly infinite plate (i.e. plate without leading-edge). Goldstein and Briggs [6] made an approximate estimate of the time required to terminate the conductive regime, i.e. the time when the effect of leading-edge appeared. But they made rather a drastic assumption that the leading-edge effect was propagated according to the velocity obtained from the solution of the doubly infinite plate. Such being the case, the limit of pure conduction has not been clear up to date.

As Ostrach [7] has noted implicitly, the limit of pure conduction is closely related to the singularity of the basic equations. In this note, the limit of pure conduction is determined without any assumption in terms of Stewartson's [8, 9] singularity which appeared in the boundary-layer equation for flow over the semi-infinite plate impulsively started from rest.

WAVE-FRONT OF LEADING-EDGE DISTURBANCE

We treat the case when the temperature of a semi-infinite plate immersed in fluid at temperature T_∞ is raised suddenly up to a higher and constant value T_w. The boundary-layer equations for unsteady laminar free convection are, in dimensionless form

$$\frac{\partial U}{\partial \tau} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \theta + \frac{\partial^2 U}{\partial Y^2} \quad (1)$$

$$\frac{\partial \theta}{\partial \tau} + U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y} = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial Y^2}. \quad (2)$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0. \quad (3)$$

The relevant boundary and initial conditions are

$$\left.\begin{array}{lll} U = V = 0,\ \theta = 1 & \text{at } Y = 0 \\ U \to 0,\ \theta \to 0 & \text{as } Y \to \infty \\ U = 0,\ \theta = 0 & \text{at } X = 0 \\ U = V = 0,\ \theta = 0 & \text{at } \tau = 0 \end{array}\right\} \quad (4)$$

Introduce a stream function ψ defined by

$$U = \partial\psi/\partial Y, \qquad V = -\partial\psi/\partial X. \quad (5)$$

It was shown by Hellums and Churchill [5] that the number of independent variables could be reduced so that $U/X^{\frac{1}{2}}$, $VX^{\frac{1}{4}}$ and θ depend only on $\tau/X^{\frac{1}{2}}$ and $Y/X^{\frac{1}{4}}$. Taking this into consideration, we introduce a function $\phi(T^*, \zeta)$.

$$\psi = 2(\sqrt{\tau})\,X^{\frac{1}{2}}\,\phi(T^*, \zeta) \quad (6)$$

where

$$T^* = \tau/X^{\frac{1}{2}}, \quad \zeta = y/2\sqrt{(\nu t)} = Y/2(\sqrt{\tau})\,[= (Y/X^{\frac{1}{4}})/2T^{*\frac{1}{2}}]. \quad (7)$$

From (5) and (6), components of velocity become

$$U = X^{\frac{1}{2}}\,\partial\phi/\partial\zeta, \quad V = X^{-\frac{1}{4}}\,T^{*\frac{1}{2}}\,(T^*\partial\phi/\partial T^* - \phi). \quad (8)$$

Substituting (8) into (1) and (2), one obtains

$$T^*\left(2 - T^*\frac{\partial\phi}{\partial\zeta}\right)\frac{\partial^2\phi}{\partial\zeta\partial T^*} = \frac{1}{2}\frac{\partial^3\phi}{\partial\zeta^3} + 2T^*\theta + \frac{\partial^2\phi}{\partial\zeta^2}\left[\zeta + T^*\left(\phi - T^*\frac{\partial\phi}{\partial T^*}\right)\right] - T^*\left(\frac{\partial\phi}{\partial\zeta}\right)^2 \quad (9)$$

$$T^*\left(2 - T^*\frac{\partial\phi}{\partial\zeta}\right)\frac{\partial\theta}{\partial T^*} = \frac{1}{2Pr}\frac{\partial^2\theta}{\partial\zeta^2} + \frac{\partial\theta}{\partial\zeta}\left[\zeta + T^*\left(\phi - T^*\frac{\partial\phi}{\partial T^*}\right)\right]. \quad (10)$$

The boundary conditions become

$$\left.\begin{array}{lll} \phi = \partial\phi/\partial\zeta = 0,\ \theta = 1 & \text{at } \zeta = 0 \\ \partial\phi/\partial\zeta \to 0,\ \theta \to 0 & \text{as } \zeta \to \infty \end{array}\right\} \quad (11)$$

The character of equations (9) and (10) may be clarified from the discussion similar to that given by Stewartson [8, 9]. Both (9) and (10) are parabolic partial differential equations of a heat-conducting type, with a coefficient of conductivity $T^*(2 - T^*U^*)$, where

$$U^*(T^*, \zeta) = \partial\phi/\partial\zeta = U/X^{\frac{1}{2}}. \quad (12)$$

Since $U^*(T^*, \zeta)$ has a maximum at some value of ζ, we express this as $U^*_{max}(T^*)$. If $2 - T^*U^*_{max}(T^*) > 0$, the coefficient of conductivity is positive, and there is no difficulty. However, if $2 - T^*U^*_{max}(T^*) < 0$, it changes sign twice through the boundary-layer, and equations (9) and (10) become of singular character. A critical value of T^* at which singularity appears first is determined by the following equation.

$$T^*U^*_{max}(T^*) = 2. \quad (13)$$

If we express the solution of (13) as T^*_{crit} (= const.), it may be said that equations (9) and (10) are regular for $T^* < T^*_{crit}$ and singular for $T^* > T^*_{crit}$.

The mathematical singularity at $T^* = T^*_{crit}$ means wave-front of disturbance propagating upward from the leading-edge of semi-infinite vertical plate, i.e. the motion of the wave-front is described by

$$\tau/X^{\frac{1}{2}} = T^*_{crit}\,(= \text{const.}). \quad (14)$$

By making use of (14), equation (13) will be derived from the the physical point of view as follows. Since the leading-edge disturbance travels with a maximum velocity of u, i.e. u_{max}, the following relation holds at the wave-front.

$$dx/dt = u_{max}(x, t).$$

It becomes, in dimensionless form

$$dX/d\tau = U_{max}(X, \tau). \quad (15)$$

On the other hand, equation (14) becomes, by differentiation

$$dX/d\tau = 2X/\tau.$$

Substituting the above equation into (15) and using the definition of T^*, one obtains equation (13) at the wave-front. Now it is clear that if $\tau < T^*_{crit}X^{\frac{1}{2}}$ (or $T^* < T^*_{crit}$), the fluid is unaware of the existence of the leading-edge and the velocity and temperature fields are the same as those for the doubly infinite plate.

LIMIT OF PURE CONDUCTION

As mentioned above, the solutions of (9) and (10) coincide with those of the doubly infinite plate for $T^* < T^*_{crit}$, which are given in [10] or [11] as follows:

$$\theta = \text{erfc}(\eta), \qquad U^* = T^*F(\eta) \quad (16)$$

where

$$F(\eta) = \begin{cases} 2\eta i \operatorname{erfc}(\eta) & \text{for } Pr = 1 \\ \dfrac{4}{1 - Pr}[i^2 \operatorname{erfc}(\eta) - i^2 \operatorname{erfc}(\eta/Pr^{\frac{1}{2}})] & \text{for } Pr \neq 1. \end{cases}$$

Since equation (16) should hold up to the limiting case ($T^* \to T^*_{crit}$), one obtains the following relation at $T^* = T^*_{crit}$.

$$U^*_{max}(T^*) = T^* F_{max} \qquad (17)$$

where F_{max} denotes maximum value of $F(\eta)$ and is a function of the Prandtl number. Substituting (17) into (13), the critical time T^*_{crit} can be obtained as

$$T^*_{crit} = (2/F_{max})^{\frac{1}{2}}. \qquad (18)$$

For a fixed point whose distance from the leading-edge is equal to X, the solutions of the doubly infinite plate, i.e. of the pure conduction hold up to the time $\tau = T^*_{crit} X^{\frac{1}{2}}$. The values of T^*_{crit} are given in Table 1 against the various Prandtl numbers as compared with the results of the earlier investigators. It may be pointed out that the approximate estimate by Goldstein and Briggs [6] agrees excellently with the exact value presented here.

Table 1. Values of critical time T^*_{crit}

Pr	Nanbu	Ref. [6]	Ref. [3]	Ref. [5]
0·001	1·496	1·51	2·21	—
0·01	1·639	1·66	2·21	—
0·1	2·031	2·07	2·28	—
0·72	2·904	—	2·68	—
0·733	2·916	—	2·69	2·4
1	3·143	3·21	2·85	—
2	3·787	—	3·37	—
10	6·423	6·54	6·10	—
100	16·39	16·6	18·1	—
1000	47·32	—	57·0	—

Local heat-transfer coefficient h in pure conduction regime is, from (16)

$$h = k/(\pi\alpha t)^{\frac{1}{2}}. \qquad (19)$$

After introducing local Nusselt number Nu_x and local Grashof number Gr_x, equation (19) is rewritten as

$$\frac{Nu_x}{Gr_x^{\frac{1}{4}}} = \left(\frac{Pr}{\pi}\right)^{\frac{1}{2}} T^{*-\frac{1}{2}}. \qquad (20)$$

Table 2 compares the steady state heat-transfer coefficient taken from Ostrach [12] with the value of the heat-transfer coefficient at the end of the pure conduction regime, i.e. at $T^* = T^*_{crit}$ for various Prandtl numbers. The difference between two is shown as "overshoot". Since the values at the end of pure conduction are smaller than those of steady state, the heat-transfer coefficient should experience a minimum before transition to steady state. This is confirmed strictly for the first time.

REFERENCES

1. A. J. Ede, *Advances in Heat Transfer*, edited by J. P. Hartnett and T. F. Irvine, Jr., Vol. 4, pp. 33–39. Academic Press, New York (1967).
2. S. Sugawara and I. Michiyoshi, The heat transfer by natural convection in the unsteady state on a vertical flat wall, Proc. 1st Japan Congr. Appl. Mech. 501–506 (1951).
3. R. Siegel, Transient free convection from a vertical flat plate, *Trans. Am. Soc. Mech. Engrs* **80**, 347–359 (1958).
4. J. D. Hellums and S. W. Churchill, Computation of natural convection by finite difference methods, *Int. Dev. Heat Transfer* **5**, 985–994 (1961).
5. J. D. Hellums and S. W. Churchill, Transient and steady state, free and natural convection, numerical solutions: Part 1. The isothermal, vertical plate, *A.I.Ch.E.Jl* **8**, 690–692 (1962).
6. R. J. Goldstein and D. G. Briggs, Transient free convection about vertical plates and circular cylinders, *J. Heat Transfer* **86**, 490–500 (1964).

Table 2. Values of heat-transfer coefficient $Nu_x/Gr_x^{1/4}$

Pr	End of pure conduction	Steady state [12]	Overshoot (%)
0·01	0·04407	0·05742	23·2
0·72	0·2809	0·3568	21·3
0·733	0·2829	0·3592	21·2
1	0·3182	0·4010	20·6
2	0·4100	0·5066	19·1
10	0·7040	0·8269	14·9
100	1·394	1·549	10·0
1000	2·594	2·804	7·5

7. S. OSTRACH, *Theory of Laminar Flows*, edited by F. K. MOORE, p. 669. Princeton Univ. Press, Princeton, N.J. (1964).
8. K. STEWARTSON, On the impulsive motion of a flat plate in a viscous fluid, *Q. J. Mech. Appl. Math.* **4**, 182–198 (1951).
9. K. STEWARTSON, *Advances in Applied Mechanics*, edited by H. L. DRYDEN and TH. VON KÁRMÁN, Vol. 6, pp. 16–24. Academic Press, New York (1960).
10. J. A. SCHETZ and R. EICHHORN, Unsteady natural convection in the vicinity of a doubly infinite vertical plate, *J. Heat Transfer* **84**, 334–338 (1962).
11. E. R. MENOLD and K. T. YANG, Asymptotic solutions for unsteady laminar free convection on a vertical plate, *J. Appl. Mech.* **29**, 124–126 (1962).
12. S. OSTRACH, An analysis of laminar free-convection flow and heat transfer about a flat plate parallel to the direction of the generating body force, NACA Rep. 1111 (1953).